ON THE THEORY OF THE HORIZON GYROCOMPASS

(K TEORII GIROGORIZONTKOMPASA)

PMM Vol.27, No.2, 1963, pp. 373-376

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(Received October 10, 1962)

The results of [1] are generalized. The equations of motion of a horizon gyrocompass are derived, taking into account the inertial terms and the vertical acceleration arising from the motion of the base. Aperiodicity conditions are derived for a gyrocompass taking inertial terms into account.

1. Let us introduce into consideration the right-hand coordinate system $O_1\xi_1\eta_1\zeta_1$ whose origin is located at the earth's center, and the axes are oriented toward fixed stars.

Let us also introduce the following right-hand coordinate systems whose common origin O is located at the gyrosphere's point of suspension: the system $O\xi\eta\zeta$ whose axes are parallel to the corresponding axes of the coordinate system $O_1\xi_1\eta_1\zeta_1$; the coordinate system $Ox_1y_1z_1$ in which the axis Ox_1 is directed eastward along the tangent to the earth's latitude and the axis Oy_1 is directed northward along the tangent to the earth's longitude; the system $Ox^\circ y^\circ z^\circ$ in which the axis Ox° is directed along the projection, onto the plane tangent to the earth at the point O, of the velocity vector of the gyrosphere's suspension point relative to system $O_1\xi_1\eta_1\zeta_1$, and the axis Oz° is directed vertically upward; the system $Oxyz$, attached to the gyrosphere, in which the axis Oy is directed along the gyrocompass' characteristic kinetic moment [moment of momentum] vector and the axis Oz is directed parallel to the axes of the casings of the gyroscopes.

The position of the system of axes $Oxyz$ with respect to the system of axes $Ox^\circ y^\circ z^\circ$, is determined by means of the three angles α, β and γ, where α is the angle of deviation of the gyrosphere axis from the azimuth, β is the angle of rise of the north end of the gyrosphere axis above the plane tangent to the earth at point O, and γ is the angle of rotation of the gyrosphere about the north-south line [1].

The table of direction cosines between the systems of axes $Oxyz$ and $Ox^{\circ}y^{\circ}z^{\circ}$ is shown here.

	x°	y°	z°
x	$\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma$	$\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$	$-\cos\beta\sin\gamma$
y	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$	$\sin\beta$
z	$\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma$	$\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$	$\cos\beta\cos\gamma$

The equations of motion of the gyrocompass relative to the coordinate system $O\xi\eta\zeta$ is wirtten in the form

$$\frac{dK_x}{dt} + \Omega_y K_z - \Omega_z K_y = M_x + L_x, \quad \frac{dK_y}{dt} + \Omega_z K_x - \Omega_x K_z = M_y + L_y$$

$$\frac{dK_z}{dt} + \Omega_x K_y - \Omega_y K_x = M_z + L_z \qquad (1.1)$$

Here K_x, K_y and K_z denote the projections onto the axes Ox, Oy and Oz, of the total kinetic moment of the gyrocompass; Ω_x, Ω_y and Ω_z denote the projections onto the axes Ox, Oy and Oz of the angular velocity of the system of axes $Oxyz$ relative to the system of axes $O_1\xi_1\eta_1\zeta_1$; M_x, M_y and M_z denote the moments with respect to the axes Ox, Oy and Oz of the external forces acting on the gyrosphere; L_x, L_y and L_z denote the moments with respect to these same axes of the inertial force due to the transfer motion of the gyrosphere as well as to the translation of the coordinate system $O\xi\eta\zeta$.

Equations (1.1) should be supplemented by yet another equation describing the motions of the gyroscopes inside the gyrosphere. By setting up for each gyroscope a moment equation, we have

$$-\frac{d}{dt}(A_1\dot{\varepsilon}) + \Omega_x B_{1y} - \Omega_y B_{1x} + M_{1z} = 0 \qquad (1.2)$$

$$\frac{d}{dt}(A_2\dot{\varepsilon}) + \Omega_x B_{2y} - \Omega_y B_{2x} + M_{2z} = 0 \qquad (1.3)$$

Here A_1 and A_2 are the moments of inertia of each gyroscope (together with its casing) relative to the axes of the casings; ε is the angle of deflection of the gyroscopes; the quantities M_{1z}, M_{2z} are the moments about the axes of the gyroscope casings; and

$$B_{1x} = -B_{2x} = B \sin \varepsilon, \qquad B_{1y} = B_{2y} = B \cos \varepsilon$$

represent the projections, respectively, onto the axes Ox and Oy of the characteristic kinetic moments of the gyroscopes. By subtracting (1.3) from (1.2) we obtain

$$-\frac{d}{dt}(A_1 + A_2)\dot{\varepsilon} - \Omega_y 2B \sin \varepsilon = N(\varepsilon) \qquad (N(\varepsilon) = M_{2z} - M_{1z}) \tag{1.4}$$

The moment $N(\varepsilon)$ can be generated by a special sensor.

Equation (1.4) is also the desired fourth equation. By assuming that the axes Ox, Oy and Oz are the principal axes of inertia of the gyrosphere at the point of suspension, we have

$$K_x = J_{xx}\Omega_x, \qquad K_y = J_{yy}\Omega_y + 2B\cos\varepsilon, \qquad K_z = J_{zz}\Omega_z + (A_2 - A_1)\dot{\varepsilon} \tag{1.5}$$

Here, $2B \cos \varepsilon$ is the characteristic kinetic moment of the gyrocompass; J_{xx}, J_{yy} and J_{zz} are the gyrosphere's principal moments of inertia relative to the axes Ox, Oy and Oz, of which the first two are, in general, functions of the angle ε.

We shall assume further that $A_1 = A_2 = A = \text{const}$, $J_{zz} = \text{const}$.

After taking (1.5) and the remarks we have made into account, equations (1.1) and (1.4) take the form

$$\begin{gathered}
\frac{d}{dt}(J_{xx}\Omega_x) + (J_{zz} - J_{yy})\,\Omega_y\Omega_z - 2B\cos\varepsilon\,\Omega_z = M_x + L \\
\frac{d}{dt}(J_{yy}\Omega_y) + (J_{xx} - J_{zz})\,\Omega_x\Omega_z + \frac{d}{dt}\,2B\cos\varepsilon = M_y + L_y \\
J_{zz}\frac{d\Omega_z}{dt} + (J_{yy} - J_{xx})\,\Omega_x\Omega_y + 2B\cos\varepsilon\Omega_x = M_z + L_z \\
-2A\ddot{\varepsilon} - \Omega_y 2B\sin\varepsilon = N(\varepsilon)
\end{gathered} \tag{1.6}$$

The expressions for Ω_x, Ω_y and Ω_z have the form [1]

$$\begin{gathered}
\Omega_x = \frac{v}{R}(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + (\Omega + \dot{\alpha})(-\cos\beta\sin\gamma) + \dot{\beta}\cos\gamma \\
\Omega_y = \frac{v}{R}\cos\alpha\cos\beta + (\Omega + \dot{\alpha})\sin\beta + \dot{\gamma} \\
\Omega_z = \frac{v}{R}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + (\Omega + \dot{\alpha})\cos\beta\cos\gamma + \dot{\beta}\sin\gamma
\end{gathered} \tag{1.7}$$

$$\left(v = \sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}\right)$$

Here v is the projection, onto the plane tangent to the earth at point O, of the velocity vector of the suspension point relative to the coordinate system $O_1\xi_1\eta_1\zeta_1$; R is the earth's radius; U is the earth's angular velocity of rotation; φ is the geocentric latitude of the ship's location; v_E and v_N are, respectively, the eastward and northward components, as projected onto the axes Ox_1 and Oy_1 of the velocity of the gyrosphere's suspension point relative to the earth's surface. Let us note also that

$$\Omega = U \sin\varphi + \frac{v_E}{R} \tan\varphi - \dot{\alpha}^* \qquad \left(\tan\alpha^* = -\frac{v_N}{RU\cos\varphi + v_E}\right) \tag{1.8}$$

From among the external forces acting on the gyrosphere, let us consider only the force of the earth's gravity, $P = mg$ (m is the gyrosphere mass and g is the free fall acceleration), which will be applied at the center of gravity c, of the gyrosphere.

Let us suppose that the force of gravity is directed parallel to the axis Oz^O, and that the gyrosphere's center of gravity has the coordinates $x_c = 0$, $y_c = 0$ and $z_c = -l$ in the system of axes $Oxyz$.

In this case

$$M_x = -mgl\sin\beta, \qquad M_y = -mgl\cos\beta\sin\gamma, \qquad M_z = 0 \tag{1.9}$$

Further

$$L_x = -m(a_z y_c - a_y z_c) = -mla_y, \quad L_y = -m(a_x z_c - a_z x_c) = mla_x \tag{1.10}$$
$$L_z = -m(a_y x_c - a_x y_c) = 0$$

Here a_x, a_y and a_z are the projections onto the axes Ox, Oy and Oz of the acceleration of the gyrosphere's suspension point relative to the coordinate system $O_1\xi_1\eta_1\zeta_1$.

Let us denote by v_{x^O}, v_{y^O} and v_{z^O} the projections onto the axes Ox^O, Oy^O and Oz^O of the velocity of the gyrosphere's suspension point relative to the coordinate system $O_1\xi_1\eta_1\zeta_1$. In accordance with our choice of the system of axes $Ox^Oy^Oz^O$, we have

$$v_{x^\circ} = v = \sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}, \qquad v_{y^\circ} = 0, \qquad v_{z^\circ} = \dot{R} \tag{1.11}$$

Denoting by Ω_{x^O}, Ω_{y^O} and Ω_{z^O} the projections onto its own axes of the angular velocity of the system of axes $Ox^Oy^Oz^O$ relative to the system of axes $O_1\xi_1\eta_1\zeta_1$, we have [1]

$$\Omega_{x^\circ} = 0, \qquad \Omega_{y^\circ} = v / R, \qquad \Omega_{z^\circ} = \Omega \tag{1.12}$$

Then, the projections onto the axes $Ox^Oy^Oz^O$ of the acceleration of the gyrosphere's suspension point relative to the coordinate system $O_1\xi_1\eta_1\zeta_1$, will have the form

$$a_{x^\circ} = \frac{dv_{x^\circ}}{dt} + \Omega_{y^\circ}v_{z^\circ} - \Omega_{z^\circ}v_{y^\circ} = \frac{dv}{dt} + \frac{v}{R}\dot{R}$$
$$a_{y^\circ} = \frac{dv_{y^\circ}}{dt} + \Omega_{z^\circ}v_{x^\circ} - \Omega_{x^\circ}v_{z^\circ} = \Omega v \qquad (1.13)$$
$$a_{z^\circ} = \frac{dv_{z^\circ}}{dt} + \Omega_{x^\circ}v_{y^\circ} - \Omega_{y^\circ}v_{x^\circ} = \ddot{R} - \frac{v^2}{R}$$

Taking the table of direction cosines and (1.13) into account, expressions (1.10) take the form

$$L_x = -ml\left[\left(\frac{dv}{dt} + \frac{v}{R}\dot{R}\right)(-\sin\alpha\cos\beta) + v\Omega\cos\alpha\cos\beta + \left(\ddot{R} - \frac{v^2}{R}\right)\sin\beta\right] \qquad (1.14)$$
$$L_y = ml\left[\left(\frac{dv}{dt} + \frac{v}{R}\dot{R}\right)(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) + \right.$$
$$\left. + v\Omega(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + \left(\ddot{R} - \frac{v^2}{R}\right)(-\cos\beta\sin\gamma)\right]$$
$$L_z = 0$$

Defining the perturbed location of the axes of the gyroscope rotors relative to the gyrosphere by the angle δ of rotation of the gyroscopes about the axes of their casings, we must set into equation (1.6)

$$\varepsilon = \varepsilon^\circ + \delta \qquad (1.15)$$

where ε° is some equilibrium value of angle ε.

In what follows we shall need the aperiodicity conditions obtained by Ishlinskii [1] for the precession theory of gyroscopic phenomenon. The latter have the form

$$2B\cos\varepsilon^\circ = mlv, \qquad N(\varepsilon^\circ) = -\frac{4B^2}{mlR}\cos\varepsilon^\circ\sin\varepsilon^\circ \qquad (1.16)$$

2. The aperiodicity conditions (1.16) turn out to be insufficient when in the investigation of the motion of a horizon gyrocompass the inertial terms are taken into account. Therefore it is of interest to obtain conditions for compensating ballistic deviations for the gyrocompass taking inertial terms into account. Let us note, however, that this implies the introduction of suitable external information.

Let us consider what conditions must be satisfied so that the axes $Oxyz$ always coincide with the axes $Ox^\circ y^\circ z^\circ$, and that the angle ε be equal to ε°, no matter how the gyrosphere's suspension point moves on the earth's surface. Let us set $\alpha = \beta = \gamma = \delta = 0$ in equations (1.6). They then take the form (the moments of inertia A_1 and A_2 are not taken into account, and moreover, we assume that $\dot{R} = 0$)

$$(J_{zz} - J_{yy}) \frac{v}{R} \Omega - 2B \cos \varepsilon^\circ \Omega = - mlv\Omega + M_x^*$$
$$\frac{d}{dt}\left(J_{yy} \frac{v}{R}\right) + \frac{d}{dt} 2B \cos \varepsilon^\circ = ml \frac{dv}{dt} + M_y^* \tag{2.1}$$
$$J_{zz} \frac{d\Omega}{dt} = M_z^*, \qquad - \frac{v}{R} 2B \sin \varepsilon^\circ = N(\varepsilon^\circ)$$

Here M_x^*, M_y^* and M_z^* are the compensating moments relative to axes Ox, Oy and Oz taking the external information into account.

Equations (2.1) are identically satisfied in two cases. In this connection we obtain the following conditions:

in the first case

$$2B \cos \varepsilon^\circ = mlv, \qquad N(\varepsilon^\circ) = - \frac{4B^2}{mlR} \cos \varepsilon^\circ \sin \varepsilon^\circ$$
$$M_x^* = \chi mlv\Omega, \qquad M_y^* = \frac{d}{dt}\left(J_{yy} \frac{v}{R}\right), \qquad M_z^* = J_{zz} \frac{d\Omega}{dt} \tag{2.2}$$

in the second case

$$2B \cos \varepsilon^\circ = mlv(1 + \chi), \qquad N(\varepsilon^\circ) = - \frac{4B^2}{mlR(1+\chi)} \cos \varepsilon^\circ \sin \varepsilon^\circ$$
$$M_x^* = 0, \qquad M_y^* = J_{zz} \frac{d}{dt} \frac{v}{R}, \qquad M_z^* = J_{zz} \frac{d\Omega}{dt} \tag{2.3}$$

Here

$$\chi = (J_{zz} - J_{yy}) \frac{\nu^2}{Pl}, \qquad \nu^2 = \frac{g}{R}$$

At the initial instant $t = 0$, let the axes $Oxyz$ and $Ox^\circ y^\circ z^\circ$ coincide, and let the initial value of angle $\varepsilon = \varepsilon(0)$ be determined by the formula

$$\cos \varepsilon(0) = \frac{mlv(0)}{2B} \qquad \left(\text{or } \cos \varepsilon(0) = \frac{mlv(0)(1+\chi)}{2B}\right) \tag{2.4}$$

Here $v(0)$ is the initial value of v.

During the whole motion, let the moment sensor generate the relation

$$N(\varepsilon) = - \frac{4B^2}{mlR} \cos \varepsilon \sin \varepsilon \qquad \left(\text{or } N(\varepsilon) = - \frac{4B^2}{mlR(1+\chi)} \cos \varepsilon \sin \varepsilon\right) \tag{2.5}$$

and moreover, let there be superimposed the compensating moments

$$M_x^* = \chi mlv\Omega, \quad M_y^* = \frac{d}{dt}\left(J_{yy} \frac{v}{R}\right), M_z^* = J_{zz} \frac{d\Omega}{dt}$$

or

$$M_x^* = 0, \qquad M_y^* = J_{zz} \frac{d}{dt} \frac{v}{R}, \qquad M_z^* = J_{zz} \frac{d\Omega}{dt} \tag{2.6}$$

Then

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0 \text{ when } t \geqslant 0$$

Condition (2.5) together with the available compensating moments automatically validate the condition

$$2B \cos \varepsilon = mlv \qquad (\text{or} \quad 2B \cos \varepsilon = mlv\,(1 + \chi)) \tag{2.7}$$

i.e. the gyroscopes themselves will "select" angle ε equal to ε°. There will be no ballistic deviations of the compass.

Thus, condition (2.2) or (2.3) is the aperiodicity condition for the gyrocompass when inertial terms are taken into account. However, if the initial conditions are satisfied with a small error, then by assuming angles α, β, γ and δ to be small, we obtain from equations (1.6), under condition (2.2) or (2.3), a linearized system which can be studied by the methods of the theory of small oscillations.

The author expresses his thanks to V.N. Koshliakov for valuable advice in carrying out this work.

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Translated by N.H.C.